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Three-dimensional wave equation for a cord and N-spiral solitary wave solution

Toru Shimizu†

Department of Physics, Faculty of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan

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Abstract. A three-dimensional wave equation which describes large deformations of a cord (thin rope) is obtained. It is shown that the wave equation has an N-solitary wave solution propagating with the same velocity and maintaining their initial shapes. In addition, the N-solitary wave solution of a factorized form is also derived through an application of one-spiral solitary wave solution, which is a time-dependent extension of the one-spiral curve of an elastic wire discussed by H Tsuru *J. Phys. Soc. Japan* **56** (1987) 2309.

1. Introduction

In 1987, Tsuru studied the problem of bending of an elastic wire in the three-dimensional space [1]. He gave an explicit expression for the one-spiral curve of an elastic wire. This expression can be easily extended to a time-dependent expression for the one-spiral solitary wave. In view of this, we have assumed that a solitary wave propagating along a cord may be found in terms of the above-mentioned expression for the one-spiral wave.

In this paper, we shall study the dynamics of a cord (thin rope) in a three-dimensional space where no external force such as gravity is exerted. We denote the curve of a cord in a three-dimensional space by $\theta(s,t)$ and $\phi(s,t)$. Here s is the arclength along the cord, θ is the angle between the directions of the z-axis and the tangent to the curve of the cord, ϕ is the angle between the directions of the x-axis and the projection of the tangent of the curve of a cord onto the xy-plane, and t is time. In section 2, we show that the equation of motion for the cord in a three-dimensional space reduces to a wave equation

$$\left[\left(\theta_t-\sqrt{\frac{T}{\rho}}\theta_s\right)^2+\left(\phi_t-\sqrt{\frac{T}{\rho}}\phi_s\right)^2\right]\left[\left(\theta_t+\sqrt{\frac{T}{\rho}}\theta_s\right)^2+\left(\phi_t+\sqrt{\frac{T}{\rho}}\phi_s\right)^2\right]=0 \qquad (1.1)$$

under the boundary condition

$$\theta(s,t)\longrightarrow 0 \qquad (\mathrm{mod}\quad 2\pi) \qquad \text{as} \quad |s|\longrightarrow\infty. \qquad (1.2)$$

Here and hereafter, subscripts s and t denote partial differentiation. We examine the properties of solitary wave solutions of wave equation (1.1). In section 3, we construct N-spiral solitary wave solution from one-spiral solitary wave solution by the superposition principle. Explicit expressions for $\theta\,(s,t)$ and $\phi\,(s,t)$ are given as factorized forms with respect to the variables (s,t). In addition, we consider the limit $\phi\to 0$ where a solitary

† Permanent address: Kurihama Research Center, Victor Company of Japan Ltd, Yokosuka, Kanagawa 239, Japan.

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wave in a plane propagates along a cord. We also construct N-loop solitary wave solution. Section 4 is devoted to discussions. The figures show theoretical curves of some one-spiral solitary wave solutions and one-loop solitary wave solution. We also refer to a periodic helical wave solution.

2. Wave equation

We shall consider the dynamics of a cord (thin rope) under the following conditions:
(i) a cord is flexible so that the force necessary for deformations is negligibly small,
(ii) stretching and contraction of a cord can be neglected,
(iii) a cord is sufficiently long and thin, and is homogeneous with the constant line density ρ,
(iv) no external force, such as gravitational force, is applied,
(v) effects of friction and dissipation are negligibly small.

We hold the end of the cord which is sufficiently far from the origin in the leftward direction; the other end ($+\infty$) is always fixed. We also fix the former end after we swing it and make waves propagate along the cord. We assume that the cord extends along the z-axis. Let s denote the arclength and $\big(x(s,t), y(s,t), z(s,t)\big)$ is the position on the cord. We make use of the polar coordinate representation so as to express components of an arbitrary vector as shown in figure 1. Accordingly, components of an arbitrary vector $\boldsymbol{r}$ ($|\boldsymbol{r}| = r$) can be expressed as

$$\boldsymbol{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) \tag{2.1}$$

where θ is the angle between the directions of the z-axis and the vector $\boldsymbol{r}$, and ϕ the angle between the directions of the x-axis and the projection of the vector $\boldsymbol{r}$ onto the xy-plane. We consider an infinitesimal portion of the cord in the interval $[s, s+\Delta s]$. Both ends of this infinitesimal portion are pulled by the tension T, and the directions of the tension T at s and $s+\Delta s$ are respectively specified by the angles (θ,ϕ) and $(\bar\theta,\bar\phi)$ of the polar coordinate representation as in (2.1). Here, we assumed the constant tension T. Since we consider only the translational and unidirectional motion of waves propagating along a cord with constant velocities under the condition that no external force is exerted on the cord. The constant tension assumption is shown to be a necessary and sufficient condition for the derivation of the translational and unidirectional wave solutions with constant velocities.

An equation of motion for an infinitesimal portion of the cord is given by

$$T\sin\bar\theta\cos\bar\phi - T\sin\theta\cos\phi = \rho\Delta s\frac{\partial^2 x}{\partial t^2} \tag{2.2a}$$

$$T\sin\bar\theta\sin\bar\phi - T\sin\theta\sin\phi = \rho\Delta s\frac{\partial^2 y}{\partial t^2} \tag{2.2b}$$

$$T\cos\bar\theta - T\cos\theta = \rho\Delta s\frac{\partial^2 z}{\partial t^2}. \tag{2.2c}$$

Taking (2.1) into account, the components of the unit tangential vectors $\boldsymbol{t}$ at s and $s+\Delta s$ are respectively given by

$$\boldsymbol{t}(s) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right) \tag{2.3a}$$

$$\boldsymbol{t}(s+\Delta s) = \big(\sin\bar\theta\cos\bar\phi, \sin\bar\theta\sin\bar\phi, \cos\bar\theta\big) = \left(\left.\frac{\partial x}{\partial s}\right|_{s+\Delta s}, \left.\frac{\partial y}{\partial s}\right|_{s+\Delta s}, \left.\frac{\partial z}{\partial s}\right|_{s+\Delta s}\right). \tag{2.3b}$$

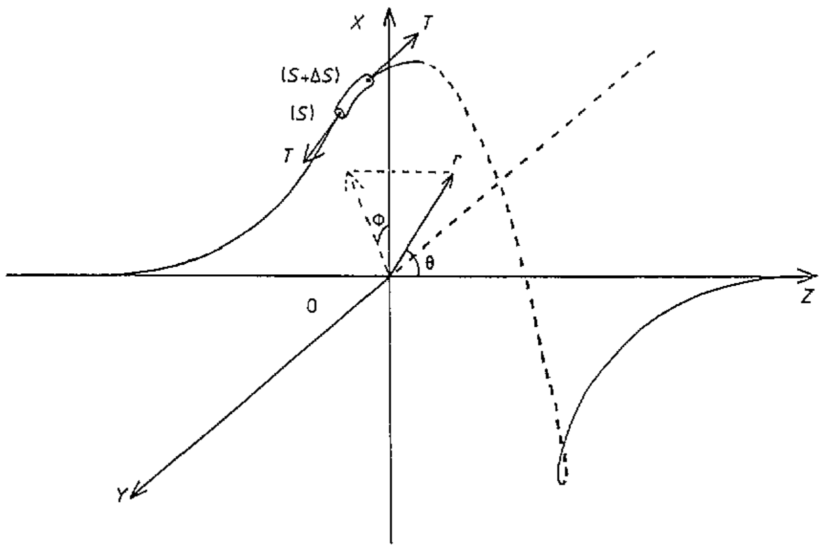

Figure 1. An infinitesimal portion of a cord in the xyz-coordinate.

Substituting equations (2.3) into (2.2) and taking the limit $\Delta s \to 0$, we obtain

$$\frac{\partial^2 x}{\partial t^2} - \kappa^4 \frac{\partial^2 x}{\partial s^2} = 0 \tag{2.4a}$$

$$\frac{\partial^2 y}{\partial t^2} - \kappa^4 \frac{\partial^2 y}{\partial s^2} = 0 \tag{2.4b}$$

$$\frac{\partial^2 z}{\partial t^2} - \kappa^4 \frac{\partial^2 z}{\partial s^2} = 0 \tag{2.4c}$$

where κ^2 is the sound velocity:

$$\kappa^4 = T/\rho. \tag{2.5}$$

We impose on equations (2.4) the boundary condition

$$\begin{cases} (z - s) \to 0 & \text{as } s \to -\infty \\ (x, y) \to (0, 0) & \text{as } |s| \to \infty. \end{cases} \tag{2.6}$$

It should be remarked that

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = 1. \tag{2.7}$$

The equation of motion for $\theta(s, t)$ and $\phi(s, t)$ may be obtained as follows. Differentiating equations (2.4) with respect to s and making use of equation (2.3a), we have

$$\theta_t^2 - \kappa^4 \theta_s^2 = \left(\theta_t - \kappa^2 \theta_s\right)\left(\theta_t + \kappa^2 \theta_s\right) = 0 \tag{2.8a}$$

and

$$\phi_t^2 - \kappa^4 \phi_s^2 = \left(\phi_t - \kappa^2 \phi_s\right)\left(\phi_t + \kappa^2 \phi_s\right) = 0 \tag{2.8b}$$

and

$$\theta_{tt} - \kappa^4 \theta_{ss} = 0 \tag{2.8c}$$

and

$$\phi_{tt} - \kappa^4 \phi_{ss} = 0 \tag{2.8d}$$

and

$$\left(\theta_t - \kappa^2 \theta_s\right)\left(\phi_t + \kappa^2 \phi_s\right) + \left(\theta_t + \kappa^2 \theta_s\right)\left(\phi_t - \kappa^2 \phi_s\right) = 0. \tag{2.8e}$$

Equations (2.8a) and (2.8b) have general solutions which describes the propagations of waves:

$$\theta(s,t) = f(s - \kappa^2 t) \qquad \text{or} \quad f(s + \kappa^2 t) \tag{2.9a}$$

$$\phi(s,t) = g(s - \kappa^2 t) \qquad \text{or} \quad g(s + \kappa^2 t) \tag{2.9b}$$

where f and g are arbitrary functions. The general solutions (2.9) of equations (2.8a) and (2.8b) are included in the general solutions of equations (2.8c) and (2.8d). Among four kinds of a pair $(\theta(s,t), \phi(s,t))$ of general solutions (2.9), only two pairs, namely

$$(\theta(s,t), \phi(s,t)) = \left(f\left(s - \kappa^2 t\right), g\left(s - \kappa^2 t\right)\right) \tag{2.10a}$$

or

$$(\theta(s,t), \phi(s,t)) = \left(f\left(s + \kappa^2 t\right), g\left(s + \kappa^2 t\right)\right) \tag{2.10b}$$

satisfy equation (2.8e). Consequently, a general solution of equations (2.8) is given by (2.10). It is easy to see that equations (2.10) are equivalent to wave equation (1.1). In order to derive translational and unidirectional wave solutions, we have assumed the tension T to be constant. This assumption leads to equations (2.10) which mean that such a wave solution propagates at a constant velocity and maintains its initial shape. Since the derivation of equations thus far holds equivalence at each step of equation reductions, the constant tension assumption must be the necessary and sufficient condition. If waves of the type described by equations (2.10) exist, then x, y, z (apart from a possible term linear in s) are functions of $(s - \kappa^2 t)$, κ^2 being constant; hence equations (2.4) imply constant tension.

We note that if we take the limit $\phi \to 0$, the analysis thus far reduces to that in the case of the wave equation for a cord in a plane. Under the limit $\phi \to 0$, equation (1.1) becomes

$$\left(\frac{\partial\theta}{\partial t} + \sqrt{\frac{T}{\rho}}\frac{\partial\theta}{\partial s}\right)\left(\frac{\partial\theta}{\partial t} - \sqrt{\frac{T}{\rho}}\frac{\partial\theta}{\partial s}\right) = 0. \tag{2.11}$$

Equation (2.11) has a general solution which describes the propagation of waves in a plane,

$$\theta(s,t) = f\left(s - \kappa^2 t\right) \qquad \text{or} \qquad f\left(s + \kappa^2 t\right) \tag{2.12}$$

where f is an arbitrary function. In dealing with two-dimensional motions (specific for a perfectly flexible rope), the tension necessary to support travelling waves is also found as part of the problem in [6].

Solutions (2.10) suggest that there exist waves of arbitrary shapes propagating in one direction only and maintaining their initial shapes [2]. On the other hand, waves propagating in the opposite directions do not co-exist.

Hereafter, we restrict the analysis to solitary wave solutions of wave equation (1.1). By the superposition principle, it is proved that a linear combination of single-solitary wave solutions with the same velocities satisfies equation (1.1). However, a linear combination of single-solitary wave solutions with the different velocities does not satisfy equation (1.1). In general, we can prove that N-solitary waves propagate at the same velocity in one direction only. It is interesting that there exist no such two or more solitary waves propagating in the opposite directions and maintaining their shapes after collisions.

3. N-Spiral solitary wave solution

It is not so difficult to observe experimentally the propagation of a spiral solitary wave along a cord. To describe such a wave, we may use one-spiral solitary wave solution which is a time-dependent extension of one-spiral curve of an elastic wire in [1]. One-spiral solitary wave solution [1] in the dispersionless case reads

$$x(s,t) = 2c\,\mathrm{sech}\left[-a\left(s - s_1 - \kappa^2 t\right)\right]\cos\left[b\left(s - s_1 - \kappa^2 t\right)\right] \tag{3.1a}$$

$$y(s,t) = 2c\,\mathrm{sech}\left[-a\left(s - s_1 - \kappa^2 t\right)\right]\sin\left[b\left(s - s_1 - \kappa^2 t\right)\right] \tag{3.1b}$$

$$z(s,t) = s + 2c\left\{\tanh\left[-a\left(s - s_1 - \kappa^2 t\right)\right] - 1\right\} \tag{3.1c}$$

where

$$a = \sqrt{4\lambda A - C^2\alpha^2}/(2A) \tag{3.2a}$$

$$b = C\alpha/(2A) \tag{3.2b}$$

$$c = a/(a^2 + b^2). \tag{3.2c}$$

Here, A and C are elastic constants of the bending and torsional stiffness, and λ and α are arbitrary constants. From equations (3.2a) and (3.2b), we have

$$a^2 + b^2 = \lambda/A \qquad (A \text{ is a constant}). \tag{3.3}$$

Taking equation (3.3) into account, we can transform constants (a, b, c) into (τ, δ) as follows:

$$a = \tau\cos\delta = \tau\cosh(\mathrm{i}\delta) \tag{3.4a}$$

$$b = \tau\sin\delta = -\mathrm{i}\tau\sinh(\mathrm{i}\delta) \tag{3.4b}$$

$$c = \cos\delta/\tau = \cosh(\mathrm{i}\delta)/\tau. \tag{3.4c}$$

We prepare one-spiral solitary wave solutions $\theta_m(s,t)$ and $\phi_m(s,t)$ of equation (1.1) from equations (3.1) with equations (3.4) and equation (2.3a):

$$\theta_m(s,t) = 2\cos^{-1}\left[\mathrm{sech}A_m \sinh^{\frac{1}{2}}(A_m + \mathrm{i}\delta)\sinh^{\frac{1}{2}}(A_m - \mathrm{i}\delta)\right] \tag{3.5a}$$

$$= 2\sin^{-1}\left[\cosh(\mathrm{i}\delta)\mathrm{sech}A_m\right] \tag{3.5b}$$

$$\phi(s,t) = \cos^{-1}\left[\frac{\cosh(\mathrm{i}\delta)\sinh A_m\cos B_m + \mathrm{i}\sinh(\mathrm{i}\delta)\cosh A_m\sin B_m}{\sinh^{\frac{1}{2}}(A_m + \mathrm{i}\delta)\sinh^{\frac{1}{2}}(A_m - \mathrm{i}\delta)}\right] \tag{3.6a}$$

$$= \sin^{-1}\left[\frac{\cosh(\mathrm{i}\delta)\sinh A_m\sin B_m - \mathrm{i}\sinh(\mathrm{i}\delta)\cosh A_m\cos B_m}{\sinh^{\frac{1}{2}}(A_m + \mathrm{i}\delta)\sinh^{\frac{1}{2}}(A_m - \mathrm{i}\delta)}\right] \tag{3.6b}$$

where

$$A_m = -a\left(s - s_m - \kappa^2 t\right) \qquad m = 1, 2, \cdots, N \tag{3.7a}$$

$$B_m = b\left(s - s_m - \kappa^2 t\right) \qquad m = 1, 2, \cdots, N \tag{3.7b}$$

and s_m are constants. The same discussion as in section 2 leads us to a conclusion that the sum of one-spiral solitary wave solutions $(\theta_m(s,t), \phi_m(s,t))$

$$\theta(s,t) = \sum_{m=1}^{N}\theta_m(s,t) \tag{3.8a}$$

$$\phi(s,t)=\sum_{m=1}^{N}\phi_m(s,t) \tag{3.8b}$$

gives the N-spiral solitary wave solution of equation (1.1). It is interesting to introduce the following complex representation for $\theta(s,t)$ and $\phi(s,t)$:

$$\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}=\mathrm{e}^{\mathrm{i}\frac{\theta}{2}} \tag{3.9a}$$

$$\cos\phi+\mathrm{i}\sin\phi=\mathrm{e}^{\mathrm{i}\phi}. \tag{3.9b}$$

From equations (3.5), equations (3.8a) and (3.9a), we have

$$\mathrm{e}^{\mathrm{i}\theta}=\prod_{m=1}^{N}\left[\sqrt{\frac{\sinh(A_m+\mathrm{i}\delta)}{\cosh A_m}}\sqrt{\frac{\sinh(A_m-\mathrm{i}\delta)}{\cosh A_m}}+\mathrm{i}\cosh(\mathrm{i}\delta)\mathrm{sech}A_m\right]^2. \tag{3.10a}$$

From equations (3.6), (3.8b) and (3.9b), we obtain

$$\mathrm{e}^{\mathrm{i}\phi}=\prod_{m=1}^{N}\left[\sqrt{\frac{\sinh(A_m+\mathrm{i}\delta)}{\sinh(A_m-\mathrm{i}\delta)}}\mathrm{e}^{\mathrm{i}B_m}\right]. \tag{3.10b}$$

The set of equations (3.10) of a factorized form gives the N-spiral wave solution of equation (1.1).

Next, we consider the propagation of the loop solitary wave in a plane. To describe such a wave, we may use one-loop solitary wave solution ($b=0$ and $c=1/a$ in (3.1)) which is also a time-dependent extension of one-kink curve of an elastic wire in [3]. One-loop solitary wave solution [3] in the dispersionless case reads

$$x(s,t)=\frac{2}{a}\mathrm{sech}\left[-a\left(s-s_1-\kappa^2t\right)\right] \tag{3.11a}$$

$$z(s,t)=s+\frac{2}{a}\left\{\tanh\left[-a\left(s-s_1-\kappa^2t\right)\right]-1\right\} \tag{3.11b}$$

where a is a constant and κ is given by equation (2.5). From equations (3.5)($\delta=0$) we have

$$\theta(s,t)=2\cos^{-1}(\tanh A_1) \tag{3.12a}$$

$$=2\sin^{-1}(\mathrm{sech}A_1) \tag{3.12b}$$

where A_1 is given by equation (3.7a). In the same manner as N-spiral solitary wave solution, we have from equation (3.10a)

$$\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}=\frac{\prod_{m=1}^{N}(\sinh A_m+\mathrm{i})}{\prod_{m=1}^{N}\cosh A_m} \tag{3.13}$$

where A_m is given by equation (3.7a). For further discussion, we impose the following assumptions on the constants s_m:

$$s_1<s_2<\ldots<s_N \tag{3.14}$$

$$s_2-s_1=s_3-s_2=\ldots=s_N-s_{N-1}. \tag{3.15}$$

This means that spacings between neighboring loops are equal. In order to reduce equation (3.13) to a simple expression for the N-loop solitary wave solution, we consider separately the cases of N odd and of N even.

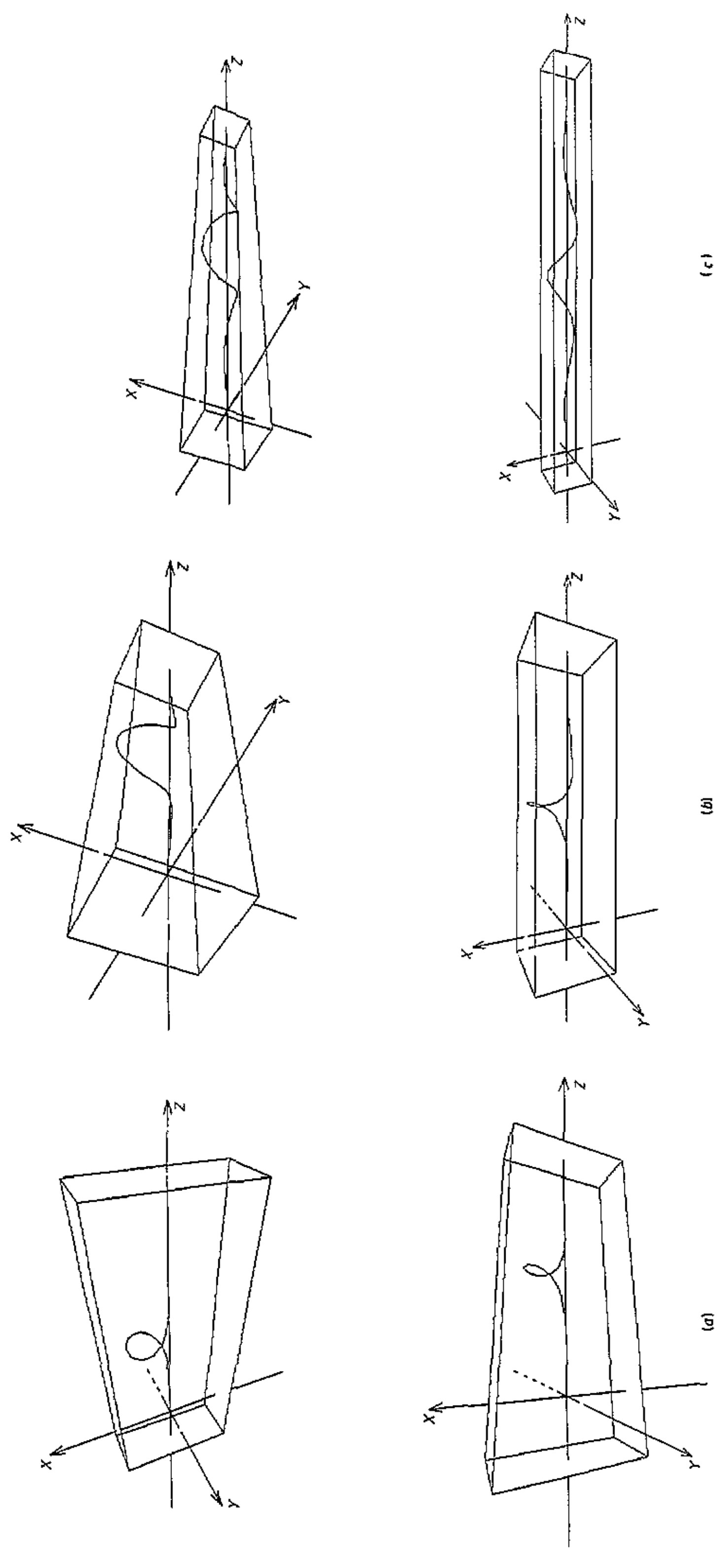

Figure 2. (*a*) A theoretical curve of one-spiral solitary wave solution ($a = 0.5$, $b = 0.2$). (*b*) A theoretical curve of one-spiral solitary wave solution ($a = 0.5$, $b = 0.4$). (*c*) A theoretical curve of one-spiral solitary wave solution ($a = 0.5$, $b = 1.0$).

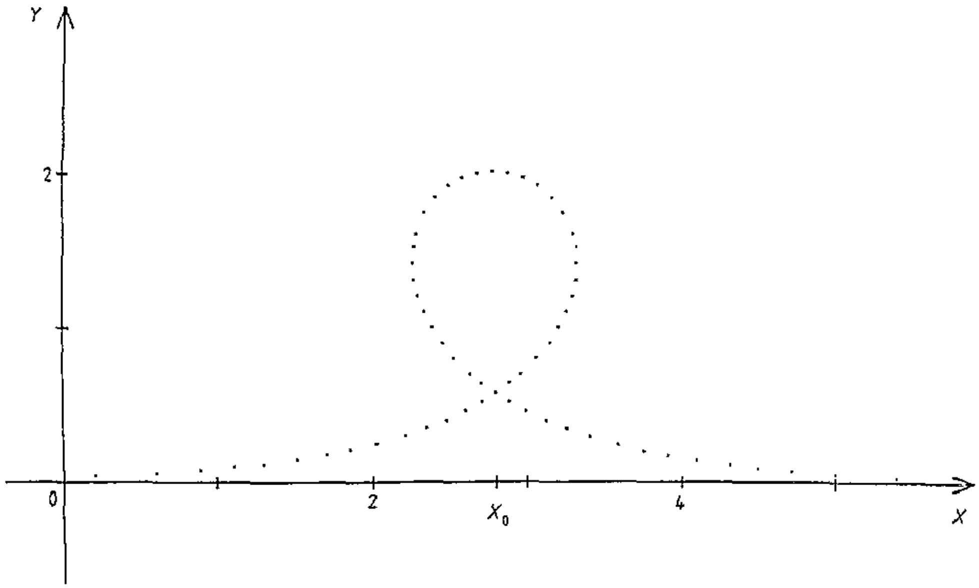

Figure 3. A theoretical curve of one-loop solitary wave solution.

(i) $N = 2n + 1$ $(n = 1, 2, \ldots)$.
Equation (3.13) can be reduced to

$$\cos\theta + \mathrm{i}\sin\theta = \prod_{m=-n}^{n}\left[\frac{\sinh A_c + \mathrm{i}\cosh(m\varphi)}{\sinh A_c - \mathrm{i}\cosh(m\varphi)}\right] \qquad n = 0, 1, 2, \ldots \tag{3.16}$$

where

$$A_1 + A_{2n+1} = A_2 + A_{2n} = \ldots = A_n + A_{n+2} = 2A_{n+1} \equiv 2A_c \tag{3.17}$$

$$\varphi = \kappa(s_2 - s_1) = \kappa(s_3 - s_2) = \ldots = \kappa(s_{2n+1} - s_{2n}). \tag{3.18}$$

We call equation (3.16) a factorized representation of the N-loop solitary wave solution of the wave equation (2.11) for $N = 2n + 1$.
(ii) $N = 2n$ $(n = 1, 2, \ldots)$.
In the same manner as in case (i), we have

$$\cos\theta + \mathrm{i}\sin\theta = \prod_{m=-(n-1)}^{n}\left[\frac{\sinh A_c + \mathrm{i}\cosh\left(m - \frac{1}{2}\right)\varphi}{\sinh A_c - \mathrm{i}\cosh\left(m - \frac{1}{2}\right)\varphi}\right] \qquad n = 1, 2, \ldots \tag{3.19}$$

where

$$A_1 + A_{2n} = A_2 + A_{2n-1} = \ldots = A_n + A_{n+1} \equiv 2A_c. \tag{3.20}$$

We call equation (3.19) a factorized representation of the N-loop solitary wave solution of the wave equation (2.11) for $N = 2n$.

4. Discussions

In this paper, we have presented the three-dimensional wave equation (1.1) for a thin rope (cord) and the N-spiral solitary wave solution. In figure 2, we show the theoretical curves of one-spiral solitary wave solution given by equations (3.1) with various values of parameters. In figure 3, we show the theoretical curve of one-loop solitary wave solution given by equations (3.11).

A periodic helical wave propagating along a cord can be observed experimentally. A periodic helical curve of an elastic wire may be expressed by the Jacobi elliptic functions [1]. The problem of investigating the relation between a periodic helical wave solution of equation (1.1) and the Jacobi elliptic functions is left for a future study.

In [1], it is remarked that the one-spiral solitary wave solution (3.1) has the same shape as the Hashimoto vortex soliton [4]. However, we point out again that our system is not a soliton system stable against collisions. It would be of interest to consider the spiral solitary wave solution from the viewpoint of motion of the curves [2, 5].

In the experiment, one-spiral or one-loop solitary wave is damped, as it propagates along a cord, mainly due to gravity effect. To eliminate the influence of external forces such as gravity, it is necessary for an ideal experiment to study the properties of spiral or loop solitary waves propagating along a cord. From this standpoint, we would like to consider such an experiment in the future.

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